

A general stochastic maximum principle for optimal control problems of forward-backward systems

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Abstract

Stochastic maximum principle of nonlinear controlled forward-backward systems, where the set of strict (classical) controls need not be convex and the diffusion coefficient depends explicitly on the variable control, is an open problem impossible to solve by the classical method of spike variation. In this paper, we introduce a new approach to solve this open problem and we establish necessary as well as sufficient conditions of optimality, in the form of global stochastic maximum principle, for two models. The first concerns the relaxed controls, who are a measure-valued processes. The second is a restriction of the first to strict control problems.

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1 Introduction

We study a stochastic control problem where the system is governed by a nonlinear forward-backward stochastic differential equation (FBSDE for short) of the type

$$\begin{cases} dx_t^v = b(t, x_t^v, v_t) dt + \sigma(t, x_t^v, v_t) dW_t, \\ x_0^v = x, \\ dy_t^v = -f(t, x_t^v, y_t^v, z_t^v, v_t) dt + z_t^v dW_t, \\ y_T^v = \varphi(x_T^v), \end{cases}$$

where b , σ , f and φ are given maps, $W = (W_t)_{t \geq 0}$ is a standard Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$, satisfying the usual conditions.

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The control variable $v = (v_t)$, called strict (classical) control, is an \mathcal{F}_t adapted process with values in some set U of \mathbb{R}^k . We denote by \mathcal{U} the class of all strict controls.

The criteria to be minimized, over the set \mathcal{U} , has the form

$$J(v) = \mathbb{E} \left[g(x_T^v) + h(y_0^v) + \int_0^T l(t, x_t^v, y_t^v, z_t^v, v_t) dt \right],$$

where g , h and l are given functions and (x_t^v, y_t^v, z_t^v) is the trajectory of the system controlled by v .

A control $u \in \mathcal{U}$ is called optimal if it satisfies

$$J(u) = \inf_{v \in \mathcal{U}} J(v).$$

The objective of this kind of stochastic control problem is to obtain the optimality conditions of controls in the form of Pontryagin stochastic maximum principle. There is many works on the subject, including Peng [42], Xu [47], Wu [46], Shi and Wu [44], Ji and Zhou [30], Bahlali and Labeled [5] and Bahlali [8]. All the previous results on stochastic maximum principle of forward-backward systems are established in the cases where the control domain is convex or uncontrolled diffusion coefficient. The general case, where the set of strict controls need not be convex and the diffusion coefficient depends explicitly on the control variable, is an open problem unsolved until now. There is no result in the literature concerning this problem and the classical way which consists to use the spike variation method on the strict controls does not lead to any result. The approach developed by Peng [41] to solve the similar case of controlled stochastic differential equations (SDEs) cannot be applied in the case of controlled FBSDEs. Indeed, since the control domain is not necessarily convex and the diffusion σ depends on the control variable, the classical way of treating such a problem would be to use the spike variation method on the strict controls and to introduce the second-order variational equation. But, the FBSDE system depends on three variables (x , y and z) and the second order expansion leads to a nonlinear problem. It is impossible to deduce then the second-order variational inequality.

In this paper, we solve this open problem by using the new approach developed by Bahlali [7]. We introduce then a bigger new class \mathcal{R} of processes by replacing the U -valued process (v_t) by a $\mathbb{P}(U)$ -valued process (q_t) , where $\mathbb{P}(U)$ is the space of probability measures on U equipped with the topology of stable convergence. This new class of processes is called relaxed controls and have a richer structure of convexity, for which the control problem becomes solvable. The main idea is to use the property of convexity of the set of relaxed controls and treat the problem with the method of convex perturbation on relaxed controls (instead of that of the spike variation on strict one). We establish then necessary and sufficient optimality conditions for relaxed controls and we derive directly the optimality conditions for strict controls from those of relaxed one.

In the relaxed model, the system is governed by the FBSDE

$$\begin{cases} dx_t^q = \int_U b(t, x_t^q, a) q_t(da) dt + \int_U \sigma(t, x_t^q, a) q_t(da) dW_t, \\ x_0^q = x, \\ dy_t^q = - \int_U f(t, x_t^q, y_t^q, z_t^q, a) q_t(da) dt + z_t^q dW_t, \\ y_T^q = \varphi(x_T^q). \end{cases}$$

The functional cost to be minimized, over the class \mathcal{R} of relaxed controls, is defined by

$$\mathcal{J}(q) = \mathbb{E} \left[g(x_T^q) + h(y_0^q) + \int_0^T \int_U l(t, x_t^q, y_t^q, z_t^q, a) q_t(da) dt \right].$$

A relaxed control μ is called optimal if it solves

$$\mathcal{J}(\mu) = \inf_{q \in \mathcal{R}} \mathcal{J}(q).$$

The relaxed control problem is a generalization of the problem of strict controls. Indeed, if $q_t(da) = \delta_{v_t}(da)$ is a Dirac measure concentrated at a single point $v_t \in U$, then we get a strict control problem as a particular case of the relaxed one.

To achieve the objective of this paper and establish necessary and sufficient optimality conditions for these two models, we proceed as follows.

Firstly, we give the optimality conditions for relaxed controls. The idea is to use the fact that the set of relaxed controls is convex. Then, we establish necessary optimality conditions by using the classical way of the convex perturbation method. More precisely, if we denote by μ an optimal relaxed control and q is an arbitrary element of \mathcal{R} , then with a sufficiently small $\theta > 0$ and for each $t \in [0, T]$, we can define a perturbed control as follows

$$\mu_t^\theta = \mu_t + \theta(q_t - \mu_t).$$

We derive the variational equation from the state equation, and the variational inequality from the inequality

$$0 \leq \mathcal{J}(\mu^\theta) - \mathcal{J}(\mu).$$

By using the fact that the coefficients b , σ , f and l are linear with respect to the relaxed control variable, necessary optimality conditions are obtained directly in the global form.

To enclose this part of the paper, we prove under minimal additional hypothesis, that these necessary optimality conditions for relaxed controls are also sufficient.

The second main result in the paper characterizes the optimality for strict control processes. It is directly derived from the above result by restricting from relaxed to strict controls. The idea is to replace the relaxed controls by a Dirac measures charging a strict controls. Thus, we reduce the set

\mathcal{R} of relaxed controls and we minimize the cost \mathcal{J} over the subset $\delta(\mathcal{U}) = \{q \in \mathcal{R} \mid q = \delta_v ; v \in \mathcal{U}\}$. Necessary optimality conditions for strict controls are then obtained directly from those of relaxed one. Finally, we prove that these necessary conditions becomes sufficient, without imposing neither the convexity of U nor that of the Hamiltonian H in v .

This paper can be also regarded as an extension of that of Bahlali [7] to the forward-backward systems. Indeed, if we consider only the forward equation, without the backward one ($y = z = f = h = 0$), we recover then exactly all the results of [7].

The paper is organized as follows. In Section 2, we formulate the strict and relaxed control problems and give the various assumptions used throughout the paper. Section 3 is devoted to study the relaxed control problems and we establish necessary as well as sufficient conditions of optimality for relaxed controls. In the last Section, we derive directly from the results of Section 3, the optimality conditions for strict controls.

Along this paper, we denote by C some positive constant, $\mathcal{M}_{n \times d}(\mathbb{R})$ the space of $n \times d$ real matrix and $\mathcal{M}_{n \times n}^d(\mathbb{R})$ the linear space of vectors $M = (M_1, \dots, M_d)$ where $M_i \in \mathcal{M}_{n \times n}(\mathbb{R})$. We use the standard calculus of inner and matrix product.

2 Formulation of the problem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$ be a filtered probability space satisfying the usual conditions, on which a d -dimensional Brownian motion $W = (W_t)_{t \geq 0}$ is defined. We assume that (\mathcal{F}_t) is the \mathcal{P} -augmentation of the natural filtration of W .

Let T be a strictly positive real number and U a non-empty set of \mathbb{R}^k .

2.1 The strict control problem

Definition 1 *An admissible strict control is an \mathcal{F}_t -adapted process $v = (v_t)$ with values in U such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} |v_t|^2 \right] < \infty.$$

We denote by \mathcal{U} the set of all admissible strict controls.

For any $v \in \mathcal{U}$, we consider the following controlled FBSDE

$$\begin{cases} dx_t^v = b(t, x_t^v, v_t) dt + \sigma(t, x_t^v, v_t) dW_t, \\ x_0^v = x, \\ dy_t^v = -f(t, x_t^v, y_t^v, z_t^v, v_t) dt + z_t^v dW_t, \\ y_T^v = \varphi(x_T^v), \end{cases} \quad (1)$$

where,

$$\begin{aligned} b &: [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n, \\ \sigma &: [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathcal{M}_{n \times d}(\mathbb{R}), \\ f &: [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{M}_{m \times d}(\mathbb{R}) \times U \longrightarrow \mathbb{R}^m, \\ \varphi &: \mathbb{R}^n \longrightarrow \mathbb{R}^m, \end{aligned}$$

and x is an n -dimensional \mathcal{F}_0 -measurable random variable such that

$$\mathbb{E}|x|^2 < \infty.$$

The criteria to be minimized is defined from \mathcal{U} into \mathbb{R} by

$$J(v) = \mathbb{E} \left[g(x_T^v) + h(y_0^v) + \int_0^T l(t, x_t^v, y_t^v, z_t^v, v_t) dt \right], \quad (2)$$

where,

$$\begin{aligned} g &: \mathbb{R}^n \longrightarrow \mathbb{R}, \\ h &: \mathbb{R}^m \longrightarrow \mathbb{R}, \\ l &: [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{M}_{m \times d}(\mathbb{R}) \times U \longrightarrow \mathbb{R}. \end{aligned}$$

A strict control u is called optimal if it satisfies

$$J(u) = \inf_{v \in \mathcal{U}} J(v). \quad (3)$$

We assume that

$$\begin{aligned} b, \sigma, f, g, h, l \text{ and } \varphi &\text{ are continuously differentiable with respect} \\ \text{to } (x, y, z), &\text{ they are bounded by } C(1 + |x| + |y| + |z| + |v|) \text{ and their} \\ \text{derivatives with respect to } (x, y, z) &\text{ are continuous in } (x, y, z, v) \\ \text{and uniformly bounded.} & \end{aligned} \quad (4)$$

Under the above hypothesis, for every $v \in U$, equation (1) has a unique strong solution and the functional cost J is well defined from \mathcal{U} into \mathbb{R} .

2.2 The relaxed model

The idea for relaxed the strict control problem defined above is to embed the set U of strict controls into a wider class which gives a more suitable topological structure. In the relaxed model, the U -valued process v is replaced by a $\mathbb{P}(U)$ -valued process q , where $\mathbb{P}(U)$ denotes the space of probability measure on U equipped with the topology of stable convergence.

Definition 2 A relaxed control $(q_t)_t$ is a $\mathbb{P}(U)$ -valued process, progressively measurable with respect to $(\mathcal{F}_t)_t$ and such that for each t , $1_{[0, t]} \cdot q$ is \mathcal{F}_t -measurable.

We denote by \mathcal{R} the set of all relaxed controls.

Remark 3 Every relaxed control q may be desintegrated as $q(dt, da) = q(t, da) dt = q_t(da) dt$, where $q_t(da)$ is a progressively measurable process with value in the set of probability measures $\mathbb{P}(U)$.

The set U is embedded into the set \mathcal{R} of relaxed process by the mapping

$$f : v \in U \longmapsto f_v(dt, da) = \delta_{v_t}(da) dt \in \mathcal{R}$$

where δ_v is the atomic measure concentrated at a single point v .

For more details on relaxed controls, see [4], [6], [7], [16], [21], [34], [37], [38].

For any $q \in \mathcal{R}$, we consider the following relaxed FBSDE

$$\begin{cases} dx_t^q = \int_U b(t, x_t^q, a) q_t(da) dt + \int_U \sigma(t, x_t^q, a) q_t(da) dW_t, \\ x_0^q = x, \\ dy_t^q = - \int_U f(t, x_t^q, y_t^q, z_t^q, a) q_t(da) dt + z_t^q dW_t, \\ y_T^q = \varphi(x_T^q). \end{cases} \quad (5)$$

The expected cost to be minimized, in the relaxed model, is defined from \mathcal{R} into \mathbb{R} by

$$\mathcal{J}(q) = \mathbb{E} \left[g(x_T^q) + h(y_0^q) + \int_0^T \int_U l(t, x_t^q, y_t^q, z_t^q, a) q_t(da) dt \right]. \quad (6)$$

A relaxed control μ is called optimal if it solves

$$\mathcal{J}(\mu) = \inf_{q \in \mathcal{R}} \mathcal{J}(q). \quad (7)$$

Remark 4 If we put

$$\begin{aligned} \bar{b}(t, x_t^q, q_t) &= \int_U b(t, x_t^q, a) q_t(da), \\ \bar{\sigma}(t, x_t^q, q_t) &= \int_U \sigma(t, x_t^q, a) q_t(da), \\ \bar{f}(t, x_t^q, y_t^q, z_t^q, a) &= \int_U f(t, x_t^q, y_t^q, z_t^q, a) q_t(da), \\ \bar{l}(t, x_t^q, y_t^q, z_t^q, a) &= \int_U l(t, x_t^q, y_t^q, z_t^q, a) q_t(da). \end{aligned}$$

Then, equation (5) becomes

$$\begin{cases} dx_t^q = \bar{b}(t, x_t^q, q_t) dt + \bar{\sigma}(t, x_t^q, q_t) dW_t, \\ x_0^q = x, \\ dy_t^q = - \bar{f}(t, x_t^q, y_t^q, z_t^q, q_t) dt + z_t^q dW_t, \\ y_T^q = \varphi(x_T^q). \end{cases}$$

With a functional cost given by

$$\mathcal{J}(q) = \mathbb{E} \left[g(x_T^q) + h(y_0^q) + \int_0^T \bar{l}(t, x_t^q, y_t^q, z_t^q, q_t) dt \right].$$

Hence, by introducing relaxed controls, we have replaced U by a larger space $\mathbb{P}(U)$. We have gained the advantage that $\mathbb{P}(U)$ is and convex. Furthermore, the new coefficients of equation (5) and the running cost are linear with respect to the relaxed control variable.

Remark 5 The coefficients $\bar{b}, \bar{\sigma}$ and \bar{f} (defined in the above remark) check respectively the same assumptions as b, σ and f . Then, under assumptions (4), $\bar{b}, \bar{\sigma}$ and \bar{f} are uniformly Lipschitz and with linear growth. Then by classical results on FBSDEs, for every $q \in \mathcal{R}$ equation (5) has a unique strong solution.

On the other hand, It is easy to see that \bar{l} checks the same assumptions as l . Then, the functional cost \mathcal{J} is well defined from \mathcal{R} into \mathbb{R} .

Remark 6 If $q_t = \delta_{v_t}$ is an atomic measure concentrated at a single point $v_t \in U$, then for each $t \in [0, T]$ we have

$$\begin{aligned} \int_U b(t, x_t^q, a) q_t(da) &= \int_U b(t, x_t^q, a) \delta_{v_t}(da) = b(t, x_t^q, v_t), \\ \int_U \sigma(t, x_t^q, a) q_t(da) &= \int_U \sigma(t, x_t^q, a) \delta_{v_t}(da) = \sigma(t, x_t^q, v_t), \\ \int_U f(t, x_t^q, y_t^q, z_t^q, a) q_t(da) &= \int_U f(t, x_t^q, y_t^q, z_t^q, a) \delta_{v_t}(da) = f(t, x_t^q, y_t^q, z_t^q, v_t), \\ \int_U l(t, x_t^q, y_t^q, z_t^q, a) q_t(da) &= \int_U l(t, x_t^q, y_t^q, z_t^q, a) \delta_{v_t}(da) = l(t, x_t^q, y_t^q, z_t^q, v_t). \end{aligned}$$

In this case $(x^q, y^q, z^q) = (x^v, y^v, z^v)$, $\mathcal{J}(q) = J(v)$ and we get a strict control problem. So the problem of strict controls $\{(1), (2), (3)\}$ is a particular case of relaxed control problem $\{(5), (6), (7)\}$.

Remark 7 The relaxed control problems studied in El Karoui et al [16] and Bahlali, Mezerdi and Djehiche [4] is different to ours, in that they relax the corresponding infinitesimal generator of the state process, which leads to a martingale problem for which the state process driven by an orthogonal martingale measure. In our setting the driving martingale measure $q_t(da) dW_t$ is however not orthogonal. See Ma and Yong [34] for more details.

3 Optimality conditions for relaxed controls

In this section, we study the problem $\{(5), (6), (7)\}$ and we establish necessary as well as sufficient conditions of optimality for relaxed controls.

3.1 Preliminary results

Since the set \mathcal{R} is convex, then the classical way to derive necessary optimality conditions for relaxed controls is to use the convex perturbation method. More precisely, let μ be an optimal relaxed control and (x^μ, y^μ, z^μ) the solution of (5) controlled by μ . Then, for each $t \in [0, T]$ we can define a perturbed relaxed control as follows

$$\mu_t^\theta = \mu_t + \theta (q_t - \mu_t),$$

where, $\theta > 0$ is sufficiently small and q is an arbitrary element of \mathcal{R} .

Denote by $(x^\theta, y^\theta, z^\theta)$ the solution of (5) associated with μ^θ .

From optimality of μ , the variational inequality will be derived from the fact that

$$0 \leq \mathcal{J}(\mu^\theta) - \mathcal{J}(\mu).$$

For this end, we need the following classical lemmas.

Lemma 8 *Under assumptions (4), we have*

$$\lim_{\theta \rightarrow 0} \left[\sup_{t \in [0, T]} \mathbb{E} |x_t^\theta - x_t^\mu|^2 \right] = 0, \quad (8)$$

$$\lim_{\theta \rightarrow 0} \left[\sup_{t \in [0, T]} \mathbb{E} |y_t^\theta - y_t^\mu|^2 \right] = 0, \quad (9)$$

$$\lim_{\theta \rightarrow 0} \mathbb{E} \int_0^T |z_t^\theta - z_t^\mu|^2 dt = 0. \quad (10)$$

Proof. (8) is proved in [7, Lemm 9, page 2085].

Let us prove (9) and (10).

Applying Itô's formula to $(y_t^\theta - y_t^\mu)^2$, we have

$$\begin{aligned} & \mathbb{E} |y_t^\theta - y_t^\mu|^2 + \mathbb{E} \int_t^T |z_s^\theta - z_s^\mu|^2 ds = \mathbb{E} |\varphi(x_T^\theta) - \varphi(x_T^\mu)|^2 \\ & + 2\mathbb{E} \int_t^T \left| (y_s^\theta - y_s^\mu) \left[\int_U f(s, x_s^\theta, y_s^\theta, z_s^\theta, a) \mu_s^\theta(da) - \int_U f(s, x_s^\mu, y_s^\mu, z_s^\mu, a) \mu_s^\mu(da) \right] \right| ds. \end{aligned}$$

From the Young formula, for every $\varepsilon > 0$, we have

$$\begin{aligned} & \mathbb{E} |y_t^\theta - y_t^\mu|^2 + \mathbb{E} \int_t^T |z_s^\theta - z_s^\mu|^2 ds \\ & \leq \mathbb{E} |\varphi(x_T^\theta) - \varphi(x_T^\mu)|^2 + \frac{1}{\varepsilon} \mathbb{E} \int_t^T |y_s^\theta - y_s^\mu|^2 ds \\ & + \varepsilon \mathbb{E} \int_t^T \left| \int_U f(s, x_s^\theta, y_s^\theta, z_s^\theta, a) \mu_s^\theta(da) - \int_U f(s, x_s^\mu, y_s^\mu, z_s^\mu, a) \mu_s^\mu(da) \right|^2 ds. \end{aligned}$$

Then,

$$\begin{aligned}
& \mathbb{E} |y_t^\theta - y_t^\mu|^2 + \mathbb{E} \int_t^T |z_s^\theta - z_s^\mu|^2 ds \\
& \leq \mathbb{E} |\varphi(x_T^\theta) - \varphi(x_T^\mu)|^2 + \frac{1}{\varepsilon} \mathbb{E} \int_t^T |y_s^\theta - y_s^\mu|^2 ds \\
& + C\varepsilon \mathbb{E} \int_t^T \left| \int_U f(s, x_s^\theta, y_s^\theta, z_s^\theta, a) \mu_s^\theta(da) - \int_U f(s, x_s^\theta, y_s^\theta, z_s^\theta, a) \mu_s(da) \right|^2 ds \\
& + C\varepsilon \mathbb{E} \int_t^T \left| \int_U f(s, x_s^\theta, y_s^\theta, z_s^\theta, a) \mu_s(da) - \int_U f(s, x_s^\mu, y_s^\theta, z_s^\theta, a) \mu_s(da) \right|^2 ds \\
& + C\varepsilon \mathbb{E} \int_t^T \left| \int_U f(s, x_s^\mu, y_s^\theta, z_s^\theta, a) \mu_s(da) - \int_U f(s, x_s^\mu, y_s^\mu, z_s^\theta, a) \mu_s(da) \right|^2 ds \\
& + C\varepsilon \mathbb{E} \int_t^T \left| \int_U f(s, x_s^\mu, y_s^\mu, z_s^\theta, a) \mu_s(da) - \int_U f(s, x_s^\mu, y_s^\mu, z_s^\mu, a) \mu_s(da) \right|^2 ds.
\end{aligned}$$

By the definition of μ_t^θ , we have

$$\begin{aligned}
& \mathbb{E} |y_t^\theta - y_t^\mu|^2 + \mathbb{E} \int_t^T |z_s^\theta - z_s^\mu|^2 ds \\
& \leq \mathbb{E} |\varphi(x_T^\theta) - \varphi(x_T)|^2 + \frac{1}{\varepsilon} \mathbb{E} \int_t^T |y_s^\theta - y_s^\mu|^2 ds \\
& + C\varepsilon \theta^2 \mathbb{E} \int_t^T \left| \int_U f(s, x_s^\theta, y_s^\theta, z_s^\theta, a) q_s(da) - \int_U f(s, x_s^\theta, y_s^\theta, z_s^\theta, a) \mu_s(da) \right|^2 ds \\
& + C\varepsilon \mathbb{E} \int_t^T \left| \int_U f(s, x_s^\theta, y_s^\theta, z_s^\theta, a) \mu_s(da) - \int_U f(s, x_s^\mu, y_s^\theta, z_s^\theta, a) \mu_s(da) \right|^2 ds \\
& + C\varepsilon \mathbb{E} \int_t^T \left| \int_U f(s, x_s^\mu, y_s^\theta, z_s^\theta, a) \mu_s(da) - \int_U f(s, x_s^\mu, y_s^\mu, z_s^\theta, a) \mu_s(da) \right|^2 ds \\
& + C\varepsilon \mathbb{E} \int_t^T \left| \int_U f(s, x_s^\mu, y_s^\mu, z_s^\theta, a) \mu_s(da) - \int_U f(s, x_s^\mu, y_s^\mu, z_s^\mu, a) \mu_s(da) \right|^2 ds.
\end{aligned}$$

Since φ and f are uniformly Lipschitz with respect to x, y, z , then

$$\begin{aligned}
\mathbb{E} |y_t^\theta - y_t^\mu|^2 + \mathbb{E} \int_t^T |z_s^\theta - z_s^\mu|^2 ds & \leq \left(\frac{1}{\varepsilon} + C\varepsilon \right) \mathbb{E} \int_t^T |y_s^\theta - y_s^\mu|^2 ds \\
& + C\varepsilon \mathbb{E} \int_t^T |z_s^\theta - z_s^\mu|^2 ds + \alpha_t^\theta,
\end{aligned} \tag{11}$$

where α_t^θ is given by

$$\alpha_t^\theta = \mathbb{E} |x_T^\theta - x_T^\mu|^2 + C\varepsilon \mathbb{E} \int_t^T |x_s^\theta - x_s^\mu|^2 ds + C\varepsilon \theta^2.$$

By (8), we have

$$\lim_{\theta \rightarrow 0} \alpha_t^\theta = 0. \quad (12)$$

Choose $\varepsilon = \frac{1}{2C}$, then (11) becomes

$$\mathbb{E} |y_t^\theta - y_t^\mu|^2 + \frac{1}{2} \mathbb{E} \int_t^T |z_s^\theta - z_s^\mu|^2 ds \leq \left(2C + \frac{1}{2}\right) \mathbb{E} \int_t^T |y_s^\theta - y_s^\mu|^2 ds + \alpha_t^\theta.$$

From the above inequality, we derive two inequalities

$$\mathbb{E} |y_t^\theta - y_t^\mu|^2 \leq \left(2C + \frac{1}{2}\right) \mathbb{E} \int_t^T |y_s^\theta - y_s^\mu|^2 ds + \alpha_t^\theta, \quad (13)$$

$$\mathbb{E} \int_t^T |z_s^\theta - z_s^\mu|^2 ds \leq (4C + 1) \mathbb{E} \int_t^T |y_s^\theta - y_s^\mu|^2 ds + 2\alpha_t^\theta. \quad (14)$$

By using (12), (13), Gronwall's lemma and Burkholder-Davis-Gundy inequality, we obtain (9). Finally, (10) is derived from (9) and (12). ■

Lemma 9 *Let \tilde{x}_t and \tilde{y}_t are respectively the solutions of the following linear equations (called variational equations)*

$$\begin{cases} d\tilde{x}_t = \int_U b_x(t, x_t^\mu, a) \mu_t(da) \tilde{x}_t dt + \int_U \sigma_x(t, x_t^\mu, a) \mu_t(da) \tilde{x}_t dW_t \\ \quad + \left[\int_U b(t, x_t^\mu, a) \mu_t(da) - \int_U b(t, x_t^\mu, a) q_t(da) \right] dt \\ \quad + \left[\int_U \sigma(t, x_t^\mu, a) \mu_t(da) - \int_U \sigma(t, x_t^\mu, a) q_t(da) \right] dW_t, \\ \tilde{x}_0 = 0. \end{cases} \quad (15)$$

$$\begin{cases} d\tilde{y}_t = - \int_U [f_x(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \tilde{x}_t + f_y(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \tilde{y}_t + f_z(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \tilde{z}_t] \mu_t(da) dt \\ \quad + \left[\int_U f(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) - \int_U f(t, x_t^\mu, y_t^\mu, z_t^\mu, a) q_t(da) \right] dt + \tilde{z}_t dW_t, \\ \tilde{y}_T = \varphi_x(x_T^\mu) \tilde{x}_T. \end{cases} \quad (16)$$

Then, the following estimations hold

$$\lim_{\theta \rightarrow 0} \mathbb{E} \left| \frac{x_t^\theta - x_t^\mu}{\theta} - \tilde{x}_t \right|^2 = 0, \quad (17)$$

$$\lim_{\theta \rightarrow 0} \mathbb{E} \left| \frac{y_t^\theta - y_t^\mu}{\theta} - \tilde{y}_t \right|^2 = 0, \quad (18)$$

$$\lim_{\theta \rightarrow 0} \mathbb{E} \int_0^T \left| \frac{z_t^\theta - z_t^\mu}{\theta} - \tilde{z}_t \right|^2 dt = 0. \quad (19)$$

Proof. For simplicity, we put

$$X_t^\theta = \frac{x_t^\theta - x_t^\mu}{\theta} - \tilde{x}_t, \quad (20)$$

$$Y_t^\theta = \frac{y_t^\theta - y_t^\mu}{\theta} - \tilde{y}_t, \quad (21)$$

$$Z_t^\theta = \frac{z_t^\theta - z_t^\mu}{\theta} - \tilde{z}_t. \quad (22)$$

$$\Lambda_t^\theta(a) = (t, x_t^\mu + \lambda\theta(X_t^\theta + \tilde{x}_t), y_t^\mu + \lambda\theta(Y_t^\theta + \tilde{y}_t), z_t^\mu + \lambda\theta(Z_t^\theta + \tilde{z}_t), a).$$

i) (17) is proved in [7, Lemma 10, Page 2086]

ii) Proof of (18) and (19).

By (21) and (22), we have the following FBSDE

$$\begin{cases} dY_t^\theta = (F_t^y Y_t^\theta dt + F_t^y Z_t^\theta - \gamma_t^\theta) dt + Z_t^\theta dW_t, \\ Y_T^\theta = \frac{\varphi(x_T^\theta) - \varphi(x_T^\mu)}{\theta} - \varphi_x(x_T^\mu) \tilde{x}_T, \end{cases}$$

where,

$$\begin{aligned} F_t^y &= - \int_0^1 \int_U f_y(\Lambda_t^\theta(a)) \mu_t(da) d\lambda, \\ F_t^z &= - \int_0^1 \int_U f_z(\Lambda_t^\theta(a)) \mu_t(da) d\lambda, \end{aligned}$$

and γ_t^θ is given by

$$\begin{aligned} \gamma_t^\theta &= \int_t^T \int_U f_x(\Lambda_s^\theta(a)) X_s^\theta \mu_s(da) ds \\ &+ \int_t^T \int_U [f_x(\Lambda_s^\theta(a)) (x_s^\theta - x_s^\mu) + f_y(\Lambda_s^\theta(a)) (y_s^\theta - y_s^\mu) + f_z(\Lambda_s^\theta(a)) (z_s^\theta - z_s^\mu)] q_s(da) ds \\ &- \int_t^T \int_U [f_x(\Lambda_s^\theta(a)) (x_s^\theta - x_s^\mu) + f_y(\Lambda_s^\theta(a)) (y_s^\theta - y_s^\mu) + f_z(\Lambda_s^\theta(a)) (z_s^\theta - z_s^\mu)] \mu_s(da) ds. \end{aligned}$$

Since f_x , f_y and f_z are continuous and bounded, then from (8), (9), (10) and (17), we have

$$\lim_{\theta \rightarrow 0} \mathbb{E} |\gamma_t^\theta|^2 = 0. \quad (23)$$

Applying Itô's formula to $(Y_t^\theta)^2$, we get

$$\mathbb{E} |Y_t^\theta|^2 + \mathbb{E} \int_t^T |Z_s^\theta|^2 ds = \mathbb{E} |Y_T^\theta|^2 + 2\mathbb{E} \int_t^T |Y_s^\theta (F_s^y Y_s^\theta + F_s^z Z_s^\theta - \gamma_s^\theta)| ds.$$

By using the Young formula, for every $\varepsilon > 0$, we have

$$\begin{aligned}\mathbb{E} |Y_t^\theta|^2 + \mathbb{E} \int_t^T |Z_s^\theta|^2 ds &\leq \mathbb{E} |Y_T^\theta|^2 + \frac{1}{\varepsilon} \mathbb{E} \int_t^T |Y_s^\theta|^2 ds + \varepsilon \mathbb{E} \int_t^T |(F_s^y Y_s^\theta + F_s^z Z_s^\theta - \gamma_s^\theta)|^2 ds \\ &\leq \mathbb{E} |Y_T^\theta|^2 + \frac{1}{\varepsilon} \mathbb{E} \int_t^T |Y_s^\theta|^2 ds + C\varepsilon \mathbb{E} \int_t^T |F_s^y Y_s^\theta|^2 ds \\ &\quad + C\varepsilon \mathbb{E} \int_t^T |F_s^z Z_s^\theta|^2 ds + C\varepsilon \mathbb{E} \int_t^T |\gamma_s^\theta|^2 ds.\end{aligned}$$

Since F_t^y and F_t^z are bounded, then

$$\mathbb{E} |Y_t^\theta|^2 + \mathbb{E} \int_t^T |Z_s^\theta|^2 ds \leq \left(\frac{1}{\varepsilon} + C\varepsilon \right) \mathbb{E} \int_t^T |Y_s^\theta|^2 ds + C\varepsilon \mathbb{E} \int_t^T |Z_s^\theta|^2 ds + \eta_t^\theta,$$

where

$$\eta_t^\theta = \mathbb{E} |Y_T^\theta|^2 + C\varepsilon \mathbb{E} \int_t^T |\gamma_s^\theta|^2 ds.$$

Choose $\varepsilon = \frac{1}{2C}$, then we have

$$\mathbb{E} |Y_t^\theta|^2 + \frac{1}{2} \mathbb{E} \int_t^T |Z_s^\theta|^2 ds \leq \left(2C + \frac{1}{2} \right) \mathbb{E} \int_t^T |Y_s^\theta|^2 ds + \eta_t^\theta.$$

From the above inequality, we deduce two inequalities

$$\mathbb{E} |Y_t^\theta|^2 \leq \left(2C + \frac{1}{2} \right) \mathbb{E} \int_t^T |Y_s^\theta|^2 ds + \eta_t^\theta, \quad (24)$$

$$\mathbb{E} \int_t^T |Z_s^\theta|^2 ds \leq (4C + 1) \mathbb{E} \int_t^T |Y_s^\theta|^2 ds + 2\eta_t^\theta. \quad (25)$$

On the other hand, we have

$$\begin{aligned}\mathbb{E} |Y_T^\theta|^2 &= \mathbb{E} \left| \tilde{y}_T - \frac{y_T^\theta - y_T^\mu}{\theta} \right|^2 \\ &= \mathbb{E} \left| \varphi_x(x_T^\mu) \tilde{x}_T - \frac{\varphi(x_T^\theta) - \varphi(x_T^\mu)}{\theta} \right|^2 \\ &\leq 2\mathbb{E} \int_0^1 |[\varphi_x(x_T^\mu) - \varphi_x(x_T^\mu + \lambda\theta(\tilde{x}_T + X_T^\theta))] \tilde{x}_T|^2 d\lambda \\ &\quad + 2\mathbb{E} \int_0^1 |\varphi_x(x_T^\mu + \lambda\theta(\tilde{x}_T + X_T^\theta)) X_T^\theta|^2 d\lambda.\end{aligned}$$

By using (17) and the fact that φ_x is continuous and bounded, we get

$$\lim_{\theta \rightarrow 0} \mathbb{E} |Y_T^\theta|^2 = 0. \quad (26)$$

From (23) and (26), we deduce that

$$\lim_{\theta \rightarrow 0} \eta_t^\theta = 0. \quad (27)$$

Finally, by using (24), (27), Gronwall's lemma and Bukholder-Davis-Gundy inequality, we obtain (18). Finally (19) is derived from (25), (27) and (18). ■

Lemma 10 *Let μ be an optimal control minimizing the functional \mathcal{J} over \mathcal{R} and $(x_t^\mu, y_t^\mu, z_t^\mu)$ the solution of (1) associated with μ . Then for any $q \in \mathcal{R}$, we have*

$$\begin{aligned} 0 &\leq \mathbb{E}[g_x(x_T^\mu) \tilde{x}_T] + \mathbb{E}[h_y(y_0^\mu) \tilde{y}_0] \\ &+ \mathbb{E} \int_0^T \left[\int_U l(t, x_t^\mu, y_t^\mu, z_t^\mu, a) q_t(da) - \int_U l(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \right] dt \\ &+ \mathbb{E} \int_0^T \left[\int_U l_x(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \tilde{x}_t + \int_U l_y(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \tilde{y}_t \right] dt \\ &+ \mathbb{E} \int_0^T \int_U l_z(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \tilde{z}_t dt. \end{aligned} \quad (28)$$

Proof. Let μ be an optimal relaxed control minimizing the cost \mathcal{J} over \mathcal{R} , then we get

$$\begin{aligned} 0 &\leq \mathbb{E}[g(x_T^\theta) - g(x_T^\mu)] + \mathbb{E}[h(y_0^\theta) - h(y_0^\mu)] \\ &+ \mathbb{E} \int_0^T \left[\int_U l(t, x_t^\theta, y_t^\theta, z_t^\theta, a) \mu_t^\theta(da) - \int_U l(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \right] dt \\ &= \mathbb{E}[g(x_T^\theta) - g(x_T^\mu)] + \mathbb{E}[h(y_0^\theta) - h(y_0^\mu)] \\ &+ \mathbb{E} \int_0^T \left[\int_U l(t, x_t^\theta, y_t^\theta, z_t^\theta, a) \mu_t^\theta(da) - \int_U l(t, x_t^\theta, y_t^\theta, z_t^\theta, a) \mu_t(da) \right] dt \\ &+ \mathbb{E} \int_0^T \left[\int_U l(t, x_t^\theta, y_t^\theta, z_t^\theta, a) \mu_t(da) - \int_U l(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \right] dt. \end{aligned}$$

From the definition of μ^θ , we get

$$\begin{aligned} 0 &\leq \mathbb{E}[g(x_T^\theta) - g(x_T^\mu)] + \mathbb{E}[h(y_0^\theta) - h(y_0^\mu)] \\ &+ \theta \mathbb{E} \int_0^T \left[\int_U l(t, x_t^\theta, y_t^\theta, z_t^\theta, a) q_t(da) - \int_U l(t, x_t^\theta, y_t^\theta, z_t^\theta, a) \mu_t(da) \right] dt \\ &+ \mathbb{E} \int_0^T \int_U [l(t, x_t^\theta, y_t^\theta, z_t^\theta, a) - l(t, x_t^\mu, y_t^\mu, z_t^\mu, a)] \mu_t(da) dt. \end{aligned}$$

Then,

$$\begin{aligned}
0 &\leq \mathbb{E} \int_0^1 [g_x(x_T^\mu + \lambda \theta (\tilde{x}_T + X_T^\theta)) \tilde{x}_T] d\lambda \\
&+ \mathbb{E} \int_0^1 [h_y(y_0^\mu + \lambda \theta (\tilde{y}_0 + Y_0^\theta)) \tilde{y}_0] d\lambda \\
&+ \mathbb{E} \int_0^T \int_U \int_0^1 [l_x(\Lambda_t^\theta(a)) \tilde{x}_t + l_y(\Lambda_t^\theta(a)) \tilde{y}_t + l_z(\Lambda_t^\theta(a)) \tilde{z}_t] \mu_t(da) d\lambda dt \\
&+ \mathbb{E} \int_0^T \left[\int_U l(t, x_t^\mu, y_t^\mu, z_t^\mu, a) q_t(da) - \int_U l(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \right] dt + \rho_t^\theta,
\end{aligned} \tag{29}$$

where ρ_t^θ is given by

$$\begin{aligned}
\rho_t^\theta &= \mathbb{E} \int_0^1 [g_x(x_T^\mu + \lambda \theta (\tilde{x}_T + X_T^\theta)) X_T^\theta] d\lambda \\
&+ \mathbb{E} \int_0^1 [h_y(y_0^\mu + \lambda \theta (\tilde{y}_0 + Y_0^\theta)) Y_0^\theta] d\lambda \\
&+ \mathbb{E} \int_0^T \int_U \int_0^1 [l_x(\Lambda_t^\theta(a)) (x_t^\theta - x_t^\mu) + l_y(\Lambda_t^\theta(a)) (y_t^\theta - y_t^\mu) + l_z(\Lambda_t^\theta(a)) (z_t^\theta - z_t^\mu)] q_t(da) d\lambda dt \\
&+ \mathbb{E} \int_0^T \int_U \int_0^1 [l_x(\Lambda_t^\theta(a)) (x_t^\theta - x_t^\mu) + l_y(\Lambda_t^\theta(a)) (y_t^\theta - y_t^\mu) + l_z(\Lambda_t^\theta(a)) (z_t^\theta - z_t^\mu)] \mu_t(da) d\lambda dt \\
&+ \mathbb{E} \int_0^T \int_U \int_0^1 [l_x(\Lambda_t^\theta(a)) X_T^\theta + l_y(\Lambda_t^\theta(a)) Y_t^\theta + l_z(\Lambda_t^\theta(a)) Z_t^\theta] \mu_t(da) d\lambda dt.
\end{aligned}$$

Since the derivatives g_x , h_y , l_x , l_y and l_z are bounded, then by using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\rho_t^\theta &\leq C \left(\mathbb{E} |X_T^\theta|^2 \right)^{1/2} + C \left(\mathbb{E} |Y_0^\theta|^2 \right)^{1/2} \\
&+ C \left(\mathbb{E} \int_0^T |x_t^\theta - x_t^\mu|^2 dt \right)^{1/2} + C \left(\mathbb{E} \int_0^T |y_t^\theta - y_t^\mu|^2 dt \right)^{1/2} + C \left(\mathbb{E} \int_0^T |z_t^\theta - z_t^\mu|^2 dt \right)^{1/2} \\
&+ C \left(\mathbb{E} \int_0^T |X_t^\theta|^2 dt \right)^{1/2} + C \left(\mathbb{E} \int_0^T |Y_t^\theta|^2 dt \right)^{1/2} + C \left(\mathbb{E} \int_0^T |Z_t^\theta|^2 dt \right)^{1/2}.
\end{aligned}$$

By using (8), (9), (10), (17), (18) and (19), we get

$$\lim_{\theta \rightarrow 0} \rho_t^\theta = 0.$$

Since g_x , h_y , l_x , l_y and l_z are continuous and bounded, the proof is completed by letting θ go to 0 in (29). ■

3.2 Necessary optimality conditions for relaxed controls

Starting from the variational inequality (28), we can now state necessary optimality conditions for the relaxed control problem $\{(5), (6), (7)\}$ in the global form.

Theorem 11 (*Necessary optimality conditions for relaxed controls*). *Let μ be an optimal relaxed control minimizing the functional \mathcal{J} over \mathcal{R} and (x^μ, y^μ, z^μ) the solution of (5) controlled by μ . Then, there exist three adapted processes (k^μ, p^μ, P^μ) , unique solution of the following FBSDE system (called adjoint equations)*

$$\begin{cases} dk_t^\mu = \mathcal{H}_y(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) dt \\ \quad + \mathcal{H}_z(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) dW_t, \\ k_0^\mu = h_y(y_0^\mu) \\ dp_t^\mu = -\mathcal{H}_x(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) dt + P_t^\mu dW_t, \\ p_T^\mu = g_x(x_T^\mu) + \varphi_x(x_T^\mu) k_T^\mu, \end{cases} \quad (30)$$

such that for every $q_t \in \mathbb{P}(U)$

$$\mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu, k_t^\mu, P_t^\mu) \leq \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, q_t, k_t^\mu, p_t^\mu, P_t^\mu), \quad ae, \quad (31)$$

where the Hamiltonian \mathcal{H} is defined from $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{M}_{m \times d}(\mathbb{R}) \times \mathbb{P}(U) \times \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{M}_{n \times d}(\mathbb{R})$ into \mathbb{R} by

$$\begin{aligned} \mathcal{H}(t, x, y, z, q, k, p, P) = & \int_U l(t, x, y, z, a) q_t(da) + p \int_U b(t, x, a) q_t(da) \\ & + P \int_U \sigma(t, x, a) q_t(da) + k \int_U f(t, x, y, z, a) q_t(da). \end{aligned}$$

Proof. Since $k_0^\mu = h_y(y_0^\mu)$ and $p_T^\mu = g_x(x_T^\mu) + \varphi_x(x_T^\mu) k_T^\mu$, then (28) becomes

$$\begin{aligned} 0 \leq & \mathbb{E}[p_T^\mu \tilde{x}_T] + \mathbb{E}[k_0^\mu \tilde{y}_0] - \mathbb{E}[\varphi_x(x_T^\mu) k_T^\mu] + \mathbb{E} \int_0^T \int_U l_x(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \tilde{x}_t \mu_t(da) dt \\ & + \mathbb{E} \int_0^T \int_U [l_y(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \tilde{y}_t + l_z(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \tilde{z}_t] \mu_t(da) dt \\ & + \mathbb{E} \int_0^T \left[\int_U l(t, x_t^\mu, y_t^\mu, z_t^\mu, a) q_t(da) - \int_U l(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \right] dt. \end{aligned} \quad (32)$$

By applying Itô's formula to $(p_t^\mu \tilde{x}_t)$ and $(k_t^\mu \tilde{y}_t)$, we have

$$\begin{aligned} \mathbb{E}[p_T^\mu \tilde{x}_T] = & -\mathbb{E} \int_0^T \left[\int_U f_x(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) k_t^\mu + \int_U l_x(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \right] \tilde{x}_t dt \\ & + \mathbb{E} \int_0^T p_t^\mu \left[\int_U b(t, x_t^\mu, a) q_t(da) - \int_U b(t, x_t^\mu, a) \mu_t(da) \right] dt \\ & + \mathbb{E} \int_0^T P_t^\mu \left[\int_U \sigma(t, x_t^\mu, a) q_t(da) - \int_U \sigma(t, x_t^\mu, a) \mu_t(da) \right] dt. \end{aligned}$$

$$\begin{aligned}
\mathbb{E}[k_0^\mu \tilde{y}_0] &= \mathbb{E}[k_T^\mu \tilde{y}_T] - \mathbb{E} \left[\int_0^T \int_U l_y(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \tilde{y}_t + \int_U f_x(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \tilde{x}_t k_t^\mu \right] dt \\
&\quad + \mathbb{E} \int_0^T k_t^\mu \left[\int_U f(t, x_t^\mu, y_t^\mu, z_t^\mu, a) q_t(da) - \int_U f(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \right] dt \\
&\quad - \mathbb{E} \int_0^T \int_U l_z(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \tilde{z}_t dt.
\end{aligned}$$

Then for every $q \in \mathcal{R}$, (32) becomes

$$0 \leq \mathbb{E} \int_0^T [\mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, q_t, k_t^\mu, p_t^\mu, P_t^\mu) - \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu)] dt.$$

Now, let $q \in \mathcal{R}$ and F be an arbitrary element of the σ -algebra \mathcal{F}_t , and set

$$\pi_t = q_t \mathbf{1}_F + \mu_t \mathbf{1}_{\Omega-F}.$$

It is obvious that π is an admissible relaxed control.

Applying the above inequality with π , we get

$$0 \leq \mathbb{E}[\mathbf{1}_F \{\mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, q_t, k_t^\mu, p_t^\mu, P_t^\mu) - \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu)\}], \forall F \in \mathcal{F}_t.$$

Which implies that

$$0 \leq \mathbb{E}[\mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, q_t, k_t^\mu, p_t^\mu, P_t^\mu) - \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) \mid \mathcal{F}_t].$$

The quantity inside the conditional expectation is \mathcal{F}_t -measurable, and thus the result follows immediately. ■

3.3 Sufficient optimality conditions for relaxed controls

In this subsection, we study when necessary optimality conditions (31) becomes sufficient. For any $q \in \mathcal{R}$, we denote by (x^q, y^q, z^q) the solution of equation (5) controlled by q .

Theorem 12 (*Sufficient optimality conditions for relaxed controls*). Assume that the functions g , h and $(x, y, z) \mapsto \mathcal{H}(t, x, y, z, q, k, p, P)$ are convex, and for any $q \in \mathcal{R}$, $y_T^q = \xi$, where ξ is an m -dimensional \mathcal{F}_T -measurable random variable such that

$$\mathbb{E}|\xi|^2 < \infty.$$

Then, μ is an optimal solution of the relaxed control problem $\{(5), (6), (7)\}$, if it satisfies (31).

Proof. Let μ be an arbitrary element of \mathcal{R} (candidate to be optimal). For any $q \in \mathcal{R}$, we have

$$\begin{aligned}
\mathcal{J}(q) - \mathcal{J}(\mu) &= \mathbb{E}[g(x_T^q) - g(x_T^\mu)] + \mathbb{E}[h(y_0^q) - h(y_0^\mu)] \\
&\quad + \mathbb{E} \int_0^T \left[\int_U l(t, x_t^q, y_t^q, z_t^q, a) q_t(da) - \int_U l(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \right] dt.
\end{aligned}$$

Since g and h are convex, then

$$\begin{aligned} g(x_T^q) - g(x_T^\mu) &\geq g_x(x_T^\mu)(x_T^q - x_T^\mu), \\ h(y_0^q) - h(y_0^\mu) &\geq h_y(y_0^\mu)(y_0^q - y_0^\mu). \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{J}(q) - \mathcal{J}(\mu) &\geq \mathbb{E}[g_x(x_T^\mu)(x_T^q - x_T^\mu)] + \mathbb{E}[h_y(y_0^\mu)(y_0^q - y_0^\mu)] \\ &\quad + \mathbb{E} \int_0^T \left[\int_U l(t, x_t^q, y_t^q, z_t^q, a) q_t(da) - \int_U l(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \right] dt. \end{aligned}$$

We remark from (30) that

$$\begin{aligned} p_T^\mu &= g_x(x_T^\mu), \\ k_0^\mu &= h_y(y_0^\mu). \end{aligned}$$

Then, we have

$$\begin{aligned} \mathcal{J}(q) - \mathcal{J}(\mu) &\geq \mathbb{E}[p_T^\mu(x_T^q - x_T^\mu)] + \mathbb{E}[k_0^\mu(y_0^q - y_0^\mu)] \\ &\quad + \mathbb{E} \int_0^T \left[\int_U l(t, x_t^q, y_t^q, z_t^q, a) q_t(da) - \int_U l(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \right] dt. \end{aligned}$$

By applying Itô's formula respectively to $p_t^\mu(x_t^q - x_t^\mu)$ and $k_t^\mu(y_t^q - y_t^\mu)$, we obtain

$$\begin{aligned} \mathbb{E}[p_T^\mu(x_T^q - x_T^\mu)] &= -\mathbb{E} \int_0^T \mathcal{H}_x(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu)(x_t^q - x_t^\mu) dt \\ &\quad + \mathbb{E} \int_0^T p_t^\mu \left[\int_U b(t, x_t^q, a) q_t(da) - \int_U b(t, x_t^\mu, a) \mu_t(da) \right] dt \\ &\quad + \mathbb{E} \int_0^T P_t^\mu \left[\int_U \sigma(t, x_t^q, a) q_t(da) - \int_U \sigma(t, x_t^\mu, a) \mu_t(da) \right] dt, \end{aligned}$$

$$\begin{aligned} \mathbb{E}[k_0^\mu(y_0^q - y_0^\mu)] &= -\mathbb{E} \int_0^T \mathcal{H}_y(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu)(y_t^q - y_t^\mu) dt \\ &\quad + \mathbb{E} \int_0^T k_t^\mu \left[\int_U f(t, x_t^q, y_t^q, z_t^q, a) q_t(da) - \int_U f(t, x_t^\mu, y_t^\mu, z_t^\mu, a) \mu_t(da) \right] dt \\ &\quad - \mathbb{E} \int_0^T \mathcal{H}_z(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu)(z_t^q - z_t^\mu) dt. \end{aligned}$$

Then,

$$\begin{aligned}
& \mathcal{J}(q) - \mathcal{J}(\mu) \\
& \geq \mathbb{E} \int_0^T [\mathcal{H}(t, x_t^q, y_t^q, z_t^q, q_t, k_t^\mu, p_t^\mu, P_t^\mu) - \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu)] dt \\
& - \mathbb{E} \int_0^T \mathcal{H}_x(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) (x_t^q - x_t^\mu) dt \\
& - \mathbb{E} \int_0^T \mathcal{H}_y(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) (y_t^q - y_t^\mu) dt \\
& - \mathbb{E} \int_0^T \mathcal{H}_z(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) (z_t^q - z_t^\mu) dt.
\end{aligned} \tag{33}$$

Since \mathcal{H} is convex in (x, y, z) and linear in μ , then by using the Clarke generalized gradient of \mathcal{H} evaluated at (x_t, y_t, z_t, μ_t) and the necessary optimality conditions (31), it follows by [50, Lemmas 2.2 and 2.3] that

$$\begin{aligned}
& \mathcal{H}(t, x_t^q, y_t^q, z_t^q, q_t, k_t^\mu, p_t^\mu, P_t^\mu) - \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) \\
& \geq \mathcal{H}_x(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) (x_t^q - x_t^\mu) + \mathcal{H}_y(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) (y_t^q - y_t^\mu) \\
& + \mathcal{H}_z(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) (z_t^q - z_t^\mu).
\end{aligned}$$

Or equivalently,

$$\begin{aligned}
0 & \leq \mathcal{H}(t, x_t^q, y_t^q, z_t^q, q_t, k_t^\mu, p_t^\mu, P_t^\mu) - \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) \\
& - \mathcal{H}_x(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) (x_t^q - x_t^\mu) \\
& - \mathcal{H}_y(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) (y_t^q - y_t^\mu) \\
& - \mathcal{H}_z(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) (z_t^q - z_t^\mu).
\end{aligned}$$

Then from (33), we get

$$\mathcal{J}(q) - \mathcal{J}(\mu) \geq 0.$$

The theorem is proved. ■

4 Optimality conditions for strict controls

In this section, we study the strict control problem $\{(1), (2), (3)\}$ and from the results of section 3, we derive the optimality conditions for strict controls.

Throughout this section and in addition to the assumptions (4), we suppose that

$$U \text{ is compact.} \tag{34}$$

$$b, \sigma, f \text{ and } l \text{ are bounded.} \tag{35}$$

Consider the following subset of \mathcal{R}

$$\delta(\mathcal{U}) = \{q \in \mathcal{R} \mid q = \delta_v \text{ ; } v \in \mathcal{U}\}.$$

The set $\delta(\mathcal{U})$ is the collection of all relaxed controls in the form of Dirac measure charging a strict control.

Denote by $\delta(U)$ the action set of all relaxed controls in $\delta(\mathcal{U})$.

If $q \in \delta(\mathcal{U})$, then $q = \delta_v$ with $v \in \mathcal{U}$. In this case we have for each t , $q_t \in \delta(U)$ and $q_t = \delta_{v_t}$.

We equipped $\mathbb{P}(U)$ with the topology of stable convergence. Since U is compact, then with this topology $\mathbb{P}(U)$ is a compact metrizable space. The stable convergence is required for bounded measurable functions $f(t, a)$ such that for each fixed $t \in [0, T]$, $f(t, \cdot)$ is continuous (Instead of functions bounded and continuous with respect to the pair (t, a) for the weak topology). The space $\mathbb{P}(U)$ is equipped with its Borel σ -field, which is the smallest σ -field such that the mapping $q \mapsto \int f(s, a) q(ds, da)$ are measurable for any bounded measurable function f , continuous with respect to a . For more details, see Jacod and Memin [29] and El Karoui et al [16].

This allows us to summarize some of lemmas that we will be used in the sequel.

Lemma 13 (*Chattering Lemma*). *Let q be a predictable process with values in the space of probability measures on U . Then there exists a sequence of predictable processes $(u^n)_n$ with values in U such that*

$$dq_t^n(da) = dt\delta_{u_t^n}(da) \xrightarrow{n \rightarrow \infty} dtq_t(da) \text{ stably, } \mathcal{P} - a.s. \quad (36)$$

where $\delta_{u_t^n}$ is the Dirac measure concentrated at a single point u_t^n of U .

Proof. See El Karoui et al [16]. ■

Lemma 14 *Let q be a relaxed control and $(u^n)_n$ be a sequence of strict controls such that (36) holds. Then for any bounded measurable function $f : [0, T] \times U \rightarrow \mathbb{R}$, such that for each fixed $t \in [0, T]$, $f(t, \cdot)$ is continuous, we have*

$$\int_U f(t, a) \delta_{u_t^n}(da) \xrightarrow{n \rightarrow \infty} \int_U f(t, a) q_t(da) \text{ ; } dt - a.e \quad (37)$$

Proof. By (36) and the definition of the stable convergence (see Jacod-Memin [29, definition 1.1, page 529], we have

$$\int_0^T \int_U f(t, a) \delta_{u_t^n}(da) dt \xrightarrow{n \rightarrow \infty} \int_0^T \int_U f(t, a) q_t(da) dt.$$

Put

$$g(s, a) = 1_{[0, t]}(s) f(s, a).$$

It's clear that g is bounded, measurable and continuous with respect to a . Then

$$\int_0^T \int_U g(s, a) \delta_{u_s^n}(da) ds \xrightarrow{n \rightarrow \infty} \int_0^T \int_U g(s, a) q_s(da) ds.$$

By replacing $g(s, a)$ by its value, we have

$$\int_0^t \int_U f(s, a) \delta_{u_s^n}(da) ds \xrightarrow{n \rightarrow \infty} \int_0^t \int_U f(s, a) q_s(da) ds.$$

The set $\{(s, t) ; 0 \leq s \leq t \leq T\}$ generate $\mathcal{B}_{[0, T]}$. Then, for every $B \in \mathcal{B}_{[0, T]}$ we have

$$\int_B \int_U f(s, a) \delta_{u_s^n}(da) ds \xrightarrow{n \rightarrow \infty} \int_B \int_U f(s, a) q_s(da) ds.$$

This implies that

$$\int_U f(s, a) \delta_{u_s^n}(da) \xrightarrow{n \rightarrow \infty} \int_U f(s, a) q_s(da) , \quad dt - a.e.$$

The lemma is proved. ■

The next lemma gives the stability of the controlled FBSDE with respect to the control variable.

Lemma 15 *Let $q \in \mathcal{R}$ be a relaxed control and (x^q, y^q, z^q) the corresponding trajectory. Then there exists a sequence $(u^n)_n \subset \mathcal{U}$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{t \in [0, T]} |x_t^n - x_t^q|^2 \right] = 0, \quad (38)$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{t \in [0, T]} |y_t^n - y_t^q|^2 \right] = 0, \quad (39)$$

$$\lim_{n \rightarrow \infty} \int_0^T \mathbb{E} |z_t^n - z_t^q|^2 dt = 0, \quad (40)$$

$$\lim_{n \rightarrow \infty} J(u^n) = \mathcal{J}(q). \quad (41)$$

where (x^n, y^n, z^n) denotes the solution of equation (1) associated with u^n .

Proof. Proof of (38). We have

$$\begin{aligned}
\mathbb{E} |x_t^n - x_t^q|^2 &\leq C \int_0^t \mathbb{E} \left| b(s, x_s^n, u_s^n) - \int_U b(s, x_s^q, a) q_s(da) \right|^2 ds \\
&\quad + C \int_0^t \mathbb{E} \left| \sigma(s, x_s^n, u_s^n) - \int_U \sigma(s, x_s^q, a) q_s(da) \right|^2 ds \\
&\leq C \int_0^t \mathbb{E} |b(s, x_s^n, u_s^n) - b(s, x_s^q, u_s^n)|^2 ds \\
&\quad + C \int_0^t \mathbb{E} \left| b(s, x_s^q, u_s^n) - \int_U b(s, x_s^q, a) q_s(da) \right|^2 ds \\
&\quad + C \int_0^t \mathbb{E} |\sigma(s, x_s^n, u_s^n) - \sigma(s, x_s^q, u_s^n)|^2 ds \\
&\quad + C \int_0^t \mathbb{E} \left| \sigma(s, x_s^q, u_s^n) - \int_U \sigma(s, x_s^q, a) q_s(da) \right|^2 ds
\end{aligned}$$

Since b and σ are uniformly Lipschitz with respect to x , then

$$\begin{aligned}
\mathbb{E} |x_t^n - x_t^q|^2 &\leq C \int_0^t \mathbb{E} |x_s^n - x_s^q|^2 ds \\
&\quad + C \int_0^t \mathbb{E} \left| b(s, x_s^q, u_s^n) - \int_U b(s, x_s^q, a) q_s(da) \right|^2 ds \\
&\quad + C \int_0^t \mathbb{E} \left| \int_U \sigma(s, x_s^q, a) \delta_{u_s^n}(da) - \int_U \sigma(s, x_s^q, a) q_s(da) \right|^2 ds
\end{aligned}$$

Since b and σ are bounded, measurable and continuous with respect to a , then by (37) and the dominated convergence theorem, the second and third terms in the right hand side of the above inequality tend to zero as n tends to infinity. We conclude then by using Gronwall's lemma and Burkholder-Davis-Gundy inequality.

ii) Proof of (39) and (40).

We have

$$\begin{cases} d(y_t^n - y_t^q) = & -[f(t, x_t^n, y_t^n, z_t^n, u_t^n) - f(t, x_t^n, y_t^q, z_t^q, u_t^n)] dt \\ & -[f(t, x_t^n, y_t^q, z_t^q, u_t^n) - f(t, x_t^q, y_t^q, z_t^q, u_t^n)] dt \\ & - \left[f(t, x_t^q, y_t^q, z_t^q, u_t^n) - \int_U f(t, x_t^q, y_t^q, z_t^q, a) q_t(da) \right] dt \\ & + (z_t^n - z_t^q) dW_t, \\ y_T^n - y_T^q = & \varphi(x_T^n) - \varphi(x_T^q). \end{cases}$$

Put

$$\begin{aligned} Y_t^n &= y_t^n - y_t^q, \\ Z_t^n &= z_t^n - z_t^q, \end{aligned}$$

and

$$\begin{aligned}
\Psi^n(t, Y_t^n, Z_t^n) = & -[f(t, x_t^n, y_t^q, z_t^q, u_t^n) - f(t, x_t^q, y_t^q, z_t^q, u_t^n)] dt \\
& - f(t, x_t^q, y_t^q, z_t^q, u_t^n) - \int_U f(t, x_t^q, y_t^q, z_t^q, a) q_t(da) \\
& - \int_0^1 f_y(t, x_t^q, y_t^q + \lambda(y_t^n - y_t^q), z_t^q + \lambda(z_t^n - z_t^q), u_t^n) Y_t^n d\lambda \\
& - \int_0^1 f_z(t, x_t^q, y_t^q + \lambda(y_t^n - y_t^q), z_t^q + \lambda(z_t^n - z_t^q), u_t^n) Z_t^n d\lambda.
\end{aligned} \tag{42}$$

Then

$$\begin{cases} dY_t^n = \Psi^n(t, Y_t^n, Z_t^n) dt + Z_t^n dW_t, \\ Y_T^n = \varphi(x_T^n) - \varphi(x_T^q). \end{cases} \tag{43}$$

The above equation is a linear BSDE with bounded coefficients, then by applying a priori estimates (see Briand et al [12]), we get

$$\begin{aligned}
\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t^n|^2 + \int_0^T |Z_t^n|^2 dt \right] & \leq C \mathbb{E} \left[|\varphi(x_T^n) - \varphi(x_T^q)|^2 + \left| \int_0^T |\Psi^n(t, 0, 0)| dt \right|^2 \right] \\
& \leq C \mathbb{E} \left[|\varphi(x_T^n) - \varphi(x_T^q)|^2 + \int_0^T |\Psi^n(t, 0, 0)|^2 dt \right].
\end{aligned}$$

From (42), we get

$$\begin{aligned}
\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t^n|^2 + \int_0^T |Z_t^n|^2 dt \right] & \leq C \mathbb{E} |\varphi(x_T^n) - \varphi(x_T^q)|^2 \\
& + C \mathbb{E} \int_0^T |f(t, x_t^n, y_t^q, z_t^q, u_t^n) - f(t, x_t^q, y_t^q, z_t^q, u_t^n)|^2 dt \\
& + C \mathbb{E} \int_0^T \left| f(t, x_t^q, y_t^q, z_t^q, u_t^n) - \int_U f(t, x_t^q, y_t^q, z_t^q, a) q_t(da) \right|^2 dt.
\end{aligned}$$

By (4), φ and f are uniformly Lipschitz with respect to x , then we get

$$\begin{aligned}
\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t^n|^2 + \int_0^T |Z_t^n|^2 dt \right] & \leq C \mathbb{E} |x_T^n - x_T^q|^2 + C \mathbb{E} \int_0^T |x_t^n - x_t^q|^2 dt \\
& + C \mathbb{E} \int_0^T \left| \int_U f(t, x_t^q, y_t^q, z_t^q, a) \delta_{u_t^n}(da) - \int_U f(t, x_t^q, y_t^q, z_t^q, a) q_t(da) \right|^2 dt.
\end{aligned}$$

By (38), the first and second terms in the right hand side of the above inequality tends to zero as n tends to infinity. Moreover, since f is bounded, measurable and continuous with respect to a , then by (37) and the dominated convergence theorem, the third term in the right hand side tends to zero as n tends to infinity. This prove (39) and (40).

iii) Proof of (41).

Since g , h and l are uniformly Lipschitz with respect to (x, y, z) , then by using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& |J(q^n) - \mathcal{J}(q)| \\
& \leq C \left(\mathbb{E} |x_T^n - x_T^q|^2 \right)^{1/2} + C \left(\mathbb{E} |y_0^n - y_0^q|^2 \right)^{1/2} \\
& + C \left(\int_0^T \mathbb{E} |x_t^n - x_t^q|^2 ds \right)^{1/2} + C \left(\int_0^T \mathbb{E} |y_t^n - y_t^q|^2 ds \right)^{1/2} + C \left(\mathbb{E} \int_0^T |z_t^n - z_t^q|^2 dt \right)^{1/2} \\
& + \left(\mathbb{E} \int_0^T \left| \int_U l(t, x_t^q, y_t^q, z_t^q, a) \delta_{u_t^n}(da) dt - \int_U l(t, x_t^q, y_t^q, z_t^q, a) q_t(da) \right|^2 dt \right)^{1/2}.
\end{aligned}$$

By (38), (39) and (40) the first five terms in the right hand side converge to zero. Furthermore, since h is bounded, measurable and continuous in a , then by (37) and the dominated convergence theorem, the sixth term in the right hand side tends to zero as n tends to infinity. This prove (41). ■

Lemma 16 *As a consequence of (41), the strict and the relaxed control problems have the same value functions. That is*

$$\inf_{v \in \mathcal{U}} J(v) = \inf_{q \in \mathcal{R}} \mathcal{J}(q). \quad (44)$$

Proof. Let $u \in \mathcal{U}$ and $\mu \in \mathcal{R}$ be respectively a strict and relaxed controls such that

$$J(u) = \inf_{v \in \mathcal{U}} J(v) \quad (45)$$

$$\mathcal{J}(\mu) = \inf_{q \in \mathcal{R}} \mathcal{J}(q). \quad (46)$$

By (46), we have

$$\mathcal{J}(\mu) \leq \mathcal{J}(q), \forall q \in \mathcal{R}.$$

Since $\delta(\mathcal{U}) \subset \mathcal{R}$, then

$$\mathcal{J}(\mu) \leq \mathcal{J}(q), \forall q \in \delta(\mathcal{U}).$$

Since $q \in \delta(\mathcal{U})$, then $q = \delta_v$, where $v \in \mathcal{U}$.

Then we get

$$\begin{cases} (x^q, y^q, z^q) = (x^v, y^v, z^v), \\ \mathcal{J}(q) = J(v). \end{cases}$$

Hence,

$$\mathcal{J}(\mu) \leq J(v), \forall v \in \mathcal{U}.$$

The control u becomes an element of \mathcal{U} , then we get

$$\mathcal{J}(\mu) \leq J(u). \quad (47)$$

On the other hand, by (45) we have

$$J(u) \leq J(v), \forall v \in \mathcal{U}. \quad (48)$$

The control μ becomes a relaxed control, then by lemma 13, there exists a sequence $(u^n)_n$ of strict controls such that

$$dt\mu_t^n(da) = dt\delta_{u_t^n}(da) \xrightarrow[n \rightarrow \infty]{} dt\mu_t(da) \text{ stably, } \mathcal{P} - a.s.$$

By (48), we get then

$$J(u) \leq J(u^n), \forall n \in \mathbb{N},$$

By using (41) and letting n go to infinity in the above inequality, we get

$$J(u) \leq \mathcal{J}(\mu). \quad (49)$$

Finally, by (47) and (49), the proof is completed. ■

To establish necessary optimality conditions for strict controls, we need the following lemma

Lemma 17 *The strict control u minimizes J over \mathcal{U} if and only if the relaxed control $\mu = \delta_u$ minimizes \mathcal{J} over \mathcal{R} .*

Proof. Suppose that u minimizes the cost J over \mathcal{U} , then

$$J(u) = \inf_{v \in \mathcal{U}} J(v).$$

By using (44), we get

$$J(u) = \inf_{q \in \mathcal{R}} \mathcal{J}(q).$$

Since $\mu = \delta_u$, then

$$\begin{cases} (x^\mu, y^\mu, z^\mu) = (x^u, y^u, z^u), \\ \mathcal{J}(\mu) = J(u), \end{cases} \quad (50)$$

This implies that

$$\mathcal{J}(\mu) = \inf_{q \in \mathcal{R}} \mathcal{J}(q).$$

Conversely, if $\mu = \delta_u$ minimize \mathcal{J} over \mathcal{R} , then

$$\mathcal{J}(\mu) = \inf_{q \in \mathcal{R}} \mathcal{J}(q).$$

From (44), we get

$$\mathcal{J}(\mu) = \inf_{v \in \mathcal{U}} J(v).$$

Since $\mu = \delta_u$, then relations (50) hold, and we obtain

$$J(u) = \inf_{v \in \mathcal{U}} J(v).$$

The proof is completed. ■

The following lemma, who will be used to establish sufficient optimality conditions for strict controls, shows that we get the results of the above lemma if we replace \mathcal{R} by $\delta(\mathcal{U})$.

Lemma 18 *The strict control u minimizes J over \mathcal{U} if and only if the relaxed control $\mu = \delta_u$ minimizes \mathcal{J} over $\delta(\mathcal{U})$.*

Proof. Let $\mu = \delta_u$ be an optimal relaxed control minimizing the cost \mathcal{J} over $\delta(\mathcal{U})$, we have then

$$\mathcal{J}(\mu) \leq \mathcal{J}(q), \quad \forall q \in \delta(\mathcal{U}).$$

Since $q \in \delta(\mathcal{U})$, then there exists $v \in \mathcal{U}$ such that $q = \delta_v$.

It is easy to see that

$$\begin{cases} (x^\mu, y^\mu, z^\mu) = (x^u, y^u, z^u), \\ (x^q, y^q, z^q) = (x^v, y^v, z^v), \\ \mathcal{J}(\mu) = J(u), \\ \mathcal{J}(q) = J(v). \end{cases} \quad (51)$$

Then, we get

$$J(u) \leq J(v), \quad \forall v \in \mathcal{U}.$$

Conversely, let u be a strict control minimizing the cost J over \mathcal{U} . Then

$$J(u) \leq J(v), \quad \forall v \in \mathcal{U}.$$

Since the controls $u, v \in \mathcal{U}$, then there exist $\mu, q \in \delta(\mathcal{U})$ such that

$$\mu = \delta_u, \quad q = \delta_v.$$

This implies that relations (51) hold. Consequently, we get

$$\mathcal{J}(\mu) \leq \mathcal{J}(q), \quad \forall q \in \delta(\mathcal{U}).$$

The lemma is proved. ■

4.1 Necessary optimality conditions for strict controls

Define the Hamiltonian H in the strict case from $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{M}_{m \times d}(\mathbb{R}) \times U \times \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{M}_{n \times d}(\mathbb{R})$ into \mathbb{R} by

$$H(t, x, y, z, v, k, p, P) = l(t, x, y, z, v) + pb(t, x, v) + P\sigma(t, x, v) + kf(t, x, y, z, v).$$

Theorem 19 (Necessary optimality conditions for strict controls). *Let u be an optimal control minimizing the functional J over \mathcal{U} and (x^u, y^u, z^u) the solution of (1) associated with u . Then, there exist three adapted processes (p^μ, P^μ, k^μ) , unique solution of the following FBSDE system (called adjoint equations)*

$$\begin{cases} dk_t^u = H_y(t, x_t^u, y_t^u, z_t^u, u_t, k_t^u, p_t^u, P_t^u) dt \\ \quad + H_z(t, x_t^u, y_t^u, z_t^u, u_t, k_t^u, p_t^u, P_t^u) dW_t, \\ k_0^u = h_y(y_0^u) \\ dp_t^u = -H_x(t, x_t^u, y_t^u, z_t^u, u_t, k_t^u, p_t^u, P_t^u) dt + P_t^u dW_t, \\ p_T^u = g_x(x_T^u) + \varphi_x(x_T^u) k_T^u, \end{cases} \quad (52)$$

such that for every $v_t \in U$

$$H(t, x_t^u, y_t^u, z_t^u, u_t, k_t^u, p_t^u, P_t^u) \leq H(t, x_t^u, y_t^u, z_t^u, v_t, k_t^u, p_t^u, P_t^u), \quad a.e., \quad a.s. \quad (53)$$

Proof. Let u be an optimal solution of the strict control problem $\{(1), (2), (3)\}$. Then, there exist $\mu \in \delta(\mathcal{U})$ such that

$$\mu = \delta_u.$$

Since u minimizes the cost J over \mathcal{U} , then by lemma 17, μ minimizes \mathcal{J} over \mathcal{R} . Hence, by the necessary optimality conditions for relaxed controls (Theorem 11), there exist three unique adapted processes (k^μ, p^μ, P^μ) , solution of the system of relaxed adjoint equations (30) such that, for every $q_t \in \mathbb{P}(U)$

$$\mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) \leq \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, q_t, k_t^\mu, p_t^\mu, P_t^\mu), \quad a.e., \quad a.s.$$

Since $\delta(U) \subset \mathbb{P}(U)$, then for every $v_t \in \delta(U)$, we get

$$\mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) \leq \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, q_t, k_t^\mu, p_t^\mu, P_t^\mu), \quad a.e., \quad a.s. \quad (54)$$

Since $q \in \delta(\mathcal{U})$, then there exist $v \in \mathcal{U}$ such that $q = \delta_v$.

We note that v is an arbitrary element of \mathcal{U} since q is arbitrary.

Now, since $\mu = \delta_u$ and $q = \delta_v$, we can easily see that

$$\begin{cases} (x^\mu, y^\mu, z^\mu) = (x^u, y^u, z^u), \\ (x^q, y^q, z^q) = (x^v, y^v, z^v), \\ (k^\mu, p^\mu, P^\mu) = (k^u, p^u, P^u), \\ \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) = H(t, x_t^u, y_t^u, z_t^u, u_t, k_t^u, p_t^u, P_t^u), \\ \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, q_t, k_t^\mu, p_t^\mu, P_t^\mu) = H(t, x_t^u, y_t^u, z_t^u, v_t, k_t^u, p_t^u, P_t^u), \end{cases} \quad (55)$$

where, the pair (p^u, P^u) and k^u are respectively the unique solutions of the system of strict adjoint equations (52).

Finally, by using (54) and (55), we can easily deduce (53). The proof is completed. ■

4.2 Sufficient optimality conditions for strict controls

Theorem 20 (*Sufficient optimality conditions for strict controls*). Assume that the functions g , and $(x, y, z) \mapsto H(t, x, y, z, q, k, p, P)$ are convex, and for any $v \in \mathcal{U}$, $y_T^v = \xi$, where ξ is an m -dimensional \mathcal{F}_T -measurable random variable such that

$$\mathbb{E} |\xi|^2 < \infty.$$

Then, u is an optimal solution of the control problem $\{(1), (2), (3)\}$, if it satisfies (53).

Proof. Let u be a strict control (candidate to be optimal) such that necessary optimality conditions for strict controls (53) hold. i.e, for every $v_t \in U$

$$H(t, x_t^u, y_t^u, z_t^u, u_t, k_t^u, p_t^u, P_t^u) \leq H(t, x_t^u, y_t^u, z_t^u, v_t, k_t^u, p_t^u, P_t^u), \text{ a.e., a.s.} \quad (56)$$

The controls u, v are elements of \mathcal{U} , then there exist $\mu, q \in \delta(\mathcal{U})$ such that

$$\begin{aligned} \mu &= \delta_u, \\ q &= \delta_v. \end{aligned}$$

This implies that relations (55) hold. Then by (56), we deduce that for every $q_t \in \delta(U)$

$$\mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, \mu_t, k_t^\mu, p_t^\mu, P_t^\mu) \leq \mathcal{H}(t, x_t^\mu, y_t^\mu, z_t^\mu, q_t, k_t^\mu, p_t^\mu, P_t^\mu), \text{ a.e., a.s.}$$

Since H is convex in (x, y, z) , it is easy to see that \mathcal{H} is convex in (x, y, z) , and since g and h are convex, then by the same proof that in theorem 12, we show that μ minimizes the cost \mathcal{J} over $\delta(\mathcal{U})$. Finally by lemma 18, we deduce that u minimizes the cost J over \mathcal{U} . The theorem is proved. ■

Remark 21 *The sufficient optimality conditions for strict controls are proved without assuming neither the convexity of U nor that of H in v .*

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